The Jacobi Stochastic Volatility model

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London-Paris Workshop on Mathematical Finance
Paris, September 29, 2016

ensiie Paris-Évry | Strasbourg

LaMME Laboratoire de Mathématiques et Modélisation d'Évry

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Stochastic volatility models

The volatility of stock price log-returns is stochastic

	Black-Scholes	Heston (affine SVJD)
volatility	constant	stochastic $\in \mathbb{R}_+$
calls and puts	closed-form	Fourier transform
exotic options	closed-form	...

Black-Scholes model $\subset$ Jacobi model $\rightarrow$ Heston model

- **stochastic volatility** on a parametrized **compact support**
- vanilla and exotic **option prices** have a **series representation**
- **fast and accurate** price approximations

Jacobi Stochastic Volatility model

Fix $0 \leq v_{min} < v_{max}$. Define the quadratic function

$$Q(v) = \frac{(v - v_{min})(v_{max} - v)}{(\sqrt{v_{max}} - \sqrt{v_{min}})^2} \leq v$$

Jacobi Model

Stock price dynamics $S_t = e^{X_t}$ given by

$$dV_t = \kappa(\theta - V_t)\, dt + \sigma \sqrt{Q(V_t)}\, dW_{1t}$$

$$dX_t = (r - V_t/2)\, dt + \rho \sqrt{Q(V_t)}\, dW_{1t} + \sqrt{V_t - \rho^2\, Q(V_t)}\, dW_{2t} \tag{1}$$

for $\kappa, \sigma > 0$, $\theta \in [v_{min}, v_{max}]$, interest rate r, $\rho \in [-1, 1]$, and 2-dimensional BM $W = (W_1, W_2)$

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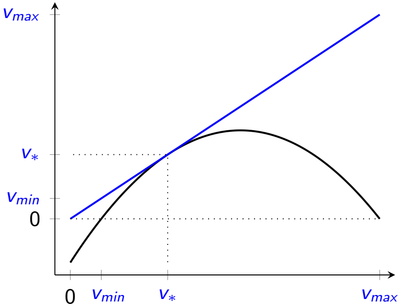
for $\kappa, \sigma > 0$, $\theta \in [v_{min}, v_{max}]$, interest rate r, $\rho \in [-1, 1]$, and 2-dimensional BM $W = (W_1, W_2)$

Remark: $\mathrm{e}^{-rt} S_t = \mathrm{e}^{-rt+X_t}$ is a martingale

Some properties

The function $Q(v)$

$v \geq Q(v)$, $v = Q(v)$ if and only if $v = \sqrt{v_{min} v_{max}}$, and $Q(v) \geq 0$ for all $v \in [v_{min}, v_{max}]$



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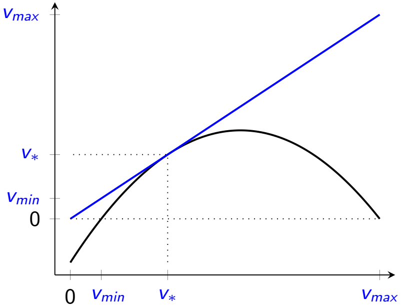

Instantaneous variance

$d\langle X, X\rangle_t = V_t \in [v_{min}, v_{max}]$ is a **Jacobi process**

Some properties (cont.)

Polynomial model

(V_t, X_t) is a polynomial diffusion – **efficient calculation of moments**

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Black-Scholes model nested

Take $v_{min} = v_{max} = \sigma^2_{\mathrm{BS}}$

Some properties (cont.)

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Take $v_{min} = v_{max} = \sigma_{\mathrm{BS}}^2$

Heston model as a limit case

If $v_{min} \to 0$ and $v_{max} \to \infty$ then (V_t, X_t) converges weakly in the path space to the Heston model

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Bounded implied volatility

Option with nonnegative BS gamma ($\Leftrightarrow$ convex payoff for Europ.)

$$\sqrt{v_{min}} \leq \sigma_{\mathrm{IV}} \leq \sqrt{v_{max}}$$

$\Rightarrow$ **Forward start option** σ_{IV} does not explode **(Jacquier and Roome 2015)**

Log-price density

We define

$$C_T = \int_0^T \left(V_t - \rho^2 Q(V_t)\right) dt$$

Theorem

Let $\epsilon < 1/(2v_{max}T)$*. If* $C_T > 0$ *then the distribution of* X_T *admits a density* $g_T(x)$ *on* $\mathbb{R}$ *that satisfies*

$$\int_{\mathbb{R}} \mathrm{e}^{\epsilon x^2} g_T(x)\, dx < \infty \qquad (2)$$

If

$$\mathbb{E}\left[{C_T}^{-1/2}\right] < \infty \qquad (3)$$

then $g_T(x)$ *and* $\mathrm{e}^{\epsilon x^2} g_T(x)$ *are uniformly bounded and continuous on* $\mathbb{R}$*. A sufficient condition for* (3) *to hold is*

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Remark: The Heston model does not satisfy (2) for any $\epsilon > 0$

A crucial corollary

Corollary

Assume (3) *holds. Then* $\ell(x) = \frac{g_T(x)}{w(x)} \in L^2_w$, *where*

$$L^2_w := \left\{ h : \int_{\mathbb{R}} |h(x)|^2 w(x)\, dx \right\}$$

and $w(x)$ *is any Gaussian density with variance* σ^2_w *satisfying*

$$\sigma^2_w > \frac{v_{max}\, T}{2} \tag{4}$$

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- (Filipovic, Mayerhofer, Schneider 2013) For the Heston model we have that $\ell(x) = \frac{g_T(x)}{w(x)} \in L^2_w$, where $w(x)$ **is a (bilateral) Gamma density**

Weighted L^2–space

The weight function

$w(x) =$ Gaussian density with mean μ_w and variance σ_w^2

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The weight function

$$w(x) = \text{Gaussian density with mean } \mu_w \text{ and variance } \sigma_w^2$$

The weighted Hilbert space

$$L_w^2 = \left\{ f(x) \mid \|f\|_w^2 = \int_{\mathbb{R}} f(x)^2 \, w(x) dx < \infty \right\}$$

which is a Hilbert space with scalar product

$$\langle f, g \rangle_w = \int_{\mathbb{R}} f(x) g(x) \, w(x) dx$$

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Orthonormal basis – Generalized Hermite polynomials

$$H_n(x) = \frac{1}{\sqrt{n!}} \mathscr{H}_n\left(\frac{x - \mu_w}{\sigma_w} \right)$$

where $\mathscr{H}_n(x)$ are the standard Hermite polynomials

Price approximation

Pricing problem

Assume that X_T has a density $g_T(x)$

$$\pi_f = \mathbb{E}[f(X_T)] = \int_{\mathbb{R}} f(x) g_T(x) dx$$

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Price series expansion

Suppose $\ell(x) = g_T(x)/w(x) \in L^2_w$ and $f(x) \in L^2_w$. Then

$$\pi_f = \langle f, \ell \rangle_w = \sum_{n \geq 0} f_n \ell_n \tag{5}$$

for the **Fourier coefficients and Hermite moments**

$$f_n = \langle f, H_n \rangle_w, \quad \ell_n = \langle \ell, H_n \rangle_w = \int_{\mathbb{R}} H_n(x) g_T(x)\, dx$$

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Price approximation

$$\pi_f \approx \pi_f^{(N)} = \sum_{n=0}^{N} f_n \ell_n = \sum_{n=0}^{N} \langle f, \ell_n H_n \rangle_w = \int_{\mathbb{R}} f(x) g_T^{(N)}(x)\, dx \tag{6}$$

Density approximation

"Gram-Charlier A expansion"

$$g_T^{(N)}(x) = w(x) \sum_{n=0}^{N} \ell_n H_n(x)$$

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Gram-Charlier expansions of prices: Jarrow and Rudd (1982), Corrado and Su (1996) ... Drimus, Necula, and Farkas (2013), Heston and Rossi (2015)...

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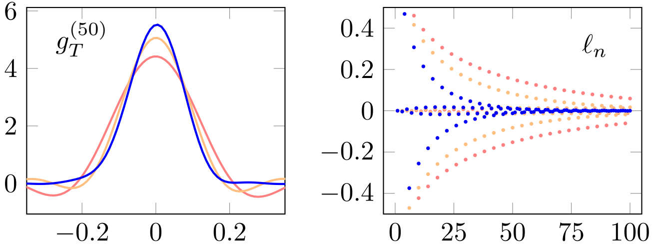


$\sigma_w \in \{1\nu, 1.5\nu, 2\nu\}$ with $\nu = \sqrt{v_{max} T/2} + \epsilon$, $T = 1/12$, $X_0 = 0$, $\kappa = 0.5$, $\theta = V_0 = (0.25)^2$, $\sigma = 0.25$, $v_{min} = (0.10)^2$, $\rho = -0.5$, and $v_{max} = 1$

European calls and puts - Fourier coefficients

Theorem

Consider the discounted payoff function for a call option with log strike k*,*

$$f(x) = \mathrm{e}^{-rT}\left(\mathrm{e}^{x} - \mathrm{e}^{k}\right)^{+}$$

Its Fourier coefficients f_n *for* $n \geq 1$ *are given by*

$$f_n = \mathrm{e}^{-rT+\mu_w}\frac{1}{\sqrt{n!}}\sigma_w I_{n-1}\left(\frac{k-\mu_w}{\sigma_w};\sigma_w\right)$$

The functions $I_n(\mu;\nu)$ *are defined recursively by*

$$I_0(\mu;\nu) = \mathrm{e}^{\frac{\nu^2}{2}}\Phi(\nu-\mu);$$
$$I_n(\mu;\nu) = \mathscr{H}_{n-1}(\mu)\mathrm{e}^{\nu\mu}\phi(\mu) + \nu I_{n-1}(\mu;\nu), \quad n \geq 1$$

where $\mathscr{H}_n(x)$ *are the standard Hermite polynomials,* $\Phi(x)$ *denotes the standard Gaussian distribution function, and* $\phi(x)$ *its density*

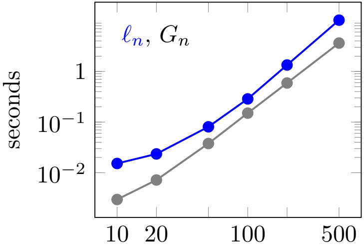
Computational cost

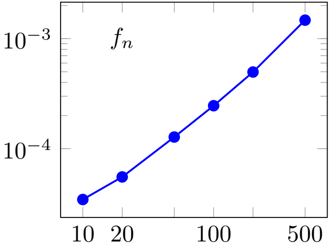
Theorem

The coefficients ℓ_n *are given by*

$$\ell_n = [h_1(V_0, X_0), \ldots, h_M(V_0, X_0)] \, \mathrm{e}^{TG_n} \, \mathbf{e}_{\pi(0,n)}, \quad 0 \le n \le N$$

where $\mathbf{e}_i$ *is the* i*–th standard basis vector in* $\mathbb{R}^M$ *and* $h_0, \ldots, h_M$ *is a basis of polynomials.* G_n *is the* $(M \times M)$*–matrix representing the infinitesimal generator of* (V_t, X_t) *on* Pol_N *– sparse matrix*





Example: Call option pricing

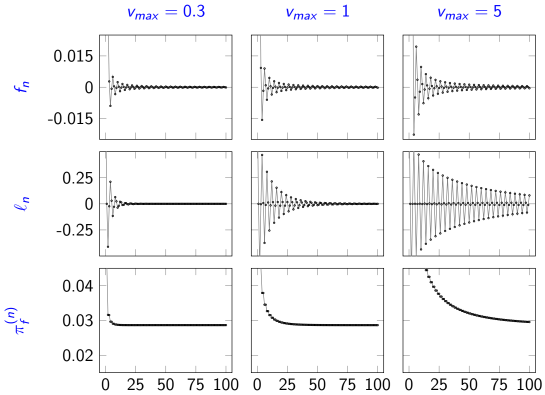


Figure: The Fourier coefficients (first row), the Hermite coefficients (second row), and the price expansion (third row) as a function of the order n. The parameters values are $T = 1/12$, $X_0 = k = 0$, $\kappa = 0.5$, $\theta = V_0 = (0.25)^2$, $\sigma = 0.25$, $v_{min} = (0.10)^2$, $\rho = -0.5$, and $v_{max} \in \{0.3, 1, 5\}$

Error bounds

Pricing error $\pi_f - \pi_f^{(N)} = \epsilon^{(N)}$

$$\left|\epsilon^{(N)}\right| = \left|\sum_{n>N} f_n \ell_n\right| \leq \sqrt{\left(\sum_{n>N} f_n^2\right)\left(\sum_{n>N} \ell_n^2\right)}$$

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Type of bounds

1. Analytic: $\ell_n^2, f_n^2 \leq C \times n^{-k}$ for some $k > 1$ and $C > 0$
2. Numeric: $\sum_{n>N} \ell_n^2 = \|\ell\|_w^2 - \sum_{n=0}^{N} \ell_n^2$

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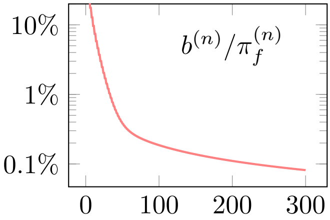
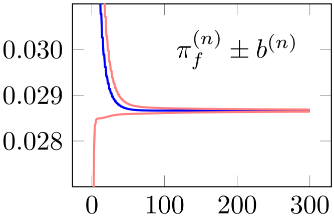
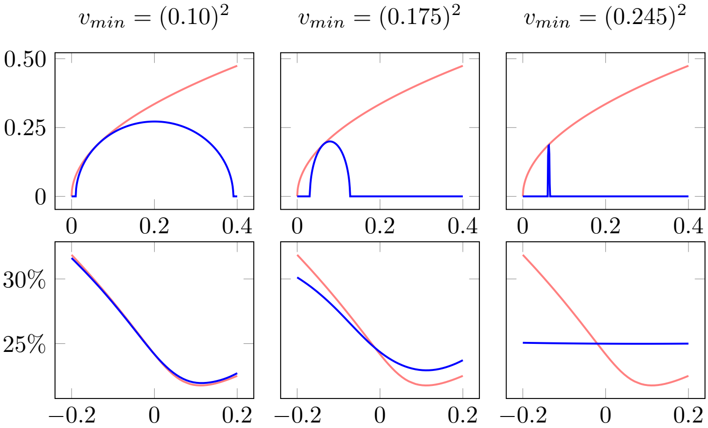




Volatility smiles - Call option

Fix $\theta = \sqrt{v_{min} v_{max}} = v_*$ and scale up v_{min}



Diffusion function $\sigma\sqrt{Q(v)}$ (1st row) and smile (2nd row)

Key corollary revisited

Log-returns density

$$Y_{t_i} = X_{t_i} - X_{t_{i-1}}$$

for $0 \leq t_0 < t_1 < t_2 < \cdots < t_n$, $Y = (Y_{t_i})$ has a density $g_{t_0,\ldots,t_n}(y)$

Weighting with Gaussians

Define $w(y) = \prod_{i=1}^{n} w_i(y_i)$ where $w_i(y_i)$ is a Gaussian density with variance $\sigma_{w_i}^2$, then $\frac{g_{t_0,\ldots,t_n}(y)}{w(y)} \in L_w^2$ if

$$\sigma_{w_i}^2 > \frac{v_{max}(t_i - t_{i-1})}{2}$$

Forward start call option

Payoff function $e^{-rt_2}(S_{t_2} - e^k S_{t_1})^+$ with $0 = t_0 < t_1 < t_2$

$$\tilde{f}(y_1, y_2) = e^{-rt_2}(e^{X_0+y_1+y_2} - e^{k+X_0+y_1})^+$$

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Fourier coefficients

$$\tilde{f}_{m_1,m_2} = \int_{\mathbb{R}^2} \tilde{f}(y) H_{m_1}(y_1) H_{m_2}(y_2) w(y) dy$$

$$= f_{m_2}^{(0,k)} \frac{\sigma_w^{m_1}}{\sqrt{m_1!}} \mathrm{e}^{X_0 - rT + \mu_{w_1} + \sigma_{w_1}^2/2}$$

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Hermite moments

$$\begin{aligned}\ell_{m_1,m_2} &= \mathbb{E}[H_{m_1}(Y_{t_1}) H_{m_2}(Y_{t_2})] \\ &= \mathbb{E}\left[H_{m_1}(Y_{t_1})\, \mathbb{E}\left[H_{m_2}(Y_{t_2}) \mid \mathscr{F}_{t_1}\right]\right]\end{aligned}$$

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Price approximation

$$\pi_{FS} = \sum_{m_1,m_2 \geq 0} \tilde{f}_{m_1,m_2} \ell_{m_1,m_2} \approx \sum_{m_1,m_2=0}^{m_1+m_2 \leq N} \tilde{f}_{m_1,m_2} \ell_{m_1,m_2} =: \pi_{FS}^{(N)}$$

Forward start options on the return

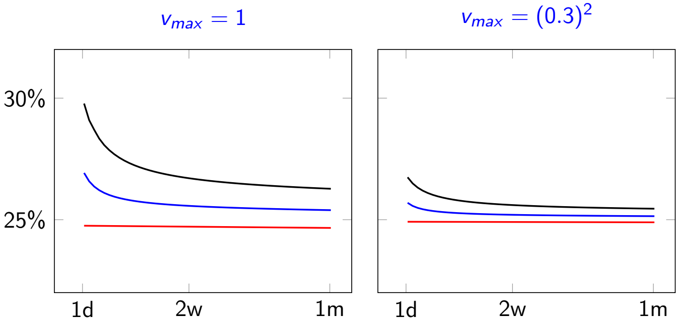


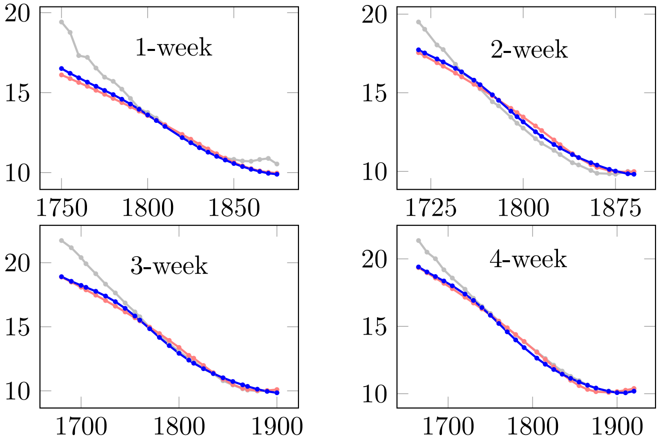
Figure: Implied volatility of a forward start option on the return with maturity $t + T$, and strikes $k = -0.10$ (black line), $k = -0.05$ (blue line), and $k = 0$ (red line) are displayed as a function of maturity T. Here $t = 1/12$, $X_0 = 0$, $\kappa = 0.5$, $V_0 = \theta = (0.25)^2$, $\sigma = 0.25$, $v_{min} = 10^{-4}$, and $\rho = -0.5$

Conclusion

- new stochastic volatility model, V_t is a **Jacobi process**
- **option price series** representation in weighted L^2_w space
  - Hermite moments (**polynomial model**)
  - Fourier coefficient (**recursive formulas**)
- computationally fast, empirically $\gtrsim$ Heston model, pricing **error bounds**
- methodology applies to **exotic option pricing**

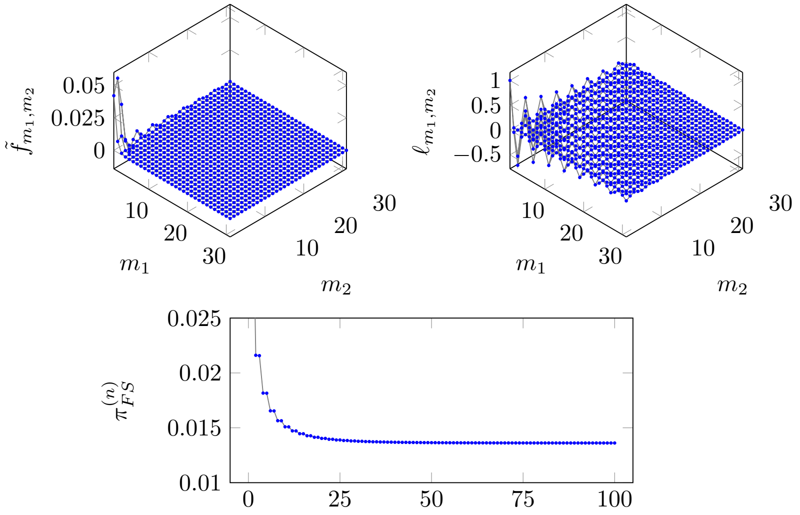
Merci beaucoup!

SPX implied volatility calibration



	$\sqrt{\theta}$	κ	σ	ρ	$\sqrt{V_0}$	$\sqrt{V_{min}}$	$\sqrt{V_{max}}$	RMSE
Jacobi	0.3660	0.7507	1.0072	-0.6057	0.1178	0.0499	0.4476	0.8461
Heston	0.3655	0.7498	0.8573	-0.6047	0.1178			0.9447

Forward start call option (cont.)



$t = 1/12$, $T - t = 1/52$, and $k = 0$